

# UNIVERSAL STRUCTURES

SAHARON SHELAH

Institute of Mathematics  
The Hebrew University of Jerusalem  
Einstein Institute of Mathematics  
Edmond J. Safra Campus, Givat Ram  
Jerusalem 91904, Israel

Department of Mathematics  
Hill Center-Busch Campus  
Rutgers, The State University of New Jersey  
110 Frelinghuysen Road  
Piscataway, NJ 08854-8019 USA

ABSTRACT. We deal with the existence of universal members in a cardinality class of abelian groups, specifically with the existence of universal members in cardinalities which are strong limit singular of countable cofinality. We then deal with the oak property (from a work of Dzamonja and the author), a property of complete first order theories, sufficient for the non-existence of universal models under suitable cardinal assumptions. We prove that it holds for the class of groups (naturally interpreted) and deal more with the existence of universals.

---

This research was partially supported by the German-Israel Foundation for Scientific Research and Development. Publication 820.

I would like to thank Alice Leonhardt for the beautiful typing.

Typeset by  $\mathcal{A}\mathcal{M}\mathcal{S}$ - $\mathcal{T}\mathcal{E}\mathcal{X}$

## ANNOTATED CONTENT

## §1 More on abelian groups

[We say more on some classes of abelian groups. We get existence and non-existence results for the existence in cardinals like  $\beth_\omega$  we use a general criterion for existence.]

## §2 Groups

[We prove that the class of groups is amenable and has the oak property (see [DjSh 710]).]

## §3 On the oak property

[We continue [DjSh 710].]

## §0 INTRODUCTION

On the existence of universal see Kojman Shelah [KjSh 409] and history there; ([KoSh 492]).

Now §1 deals with abelian groups (a case removed from [Sh 622] as it was too long; the second section deals with the class of groups continue Osvyatsov and Shelah [ShUs 789]); the third deals with the oak property continuing Dzamonja and Shelah [DjSh 710], dealing with the case of singular cardinals and later on abelian groups.

## §1 MORE ON ABELIAN GROUPS

This section originally was part of [Sh 622] (which continues [Sh 552]); earlier Kojman Shelah [KjSh 455] and [Sh 456], but as the paper was too long, it was delayed.

For me high in visibility is say  $\beth_\omega$ . There are related positive (= existence) results; see

- (a) [Sh 26, Th.3.1,p.266], where it is proved that:  
if  $\lambda$  is strong limit singular, then  
 $\{G : G \text{ a graph with } \leq \lambda \text{ nodes each of valency } < \lambda\}$  has a universal member under embedding onto induced subgraphs
- (b) by Grossberg Shelah [GrSh 174] similar results hold for quite general classes (e.g. locally finite groups) if  $\lambda$  is strong limit, of cofinality  $\aleph_0$  and is above a compact cardinal (which is quite large).

More specifically, if  $\mu$  is strong limit of cofinality  $\aleph_0$  above a compact cardinal  $\kappa$  and, e.g., the class  $\mathfrak{K}$  is the class of models of  $\tau \subseteq \mathbb{L}_{\kappa, \omega}$ ,  $|T| < \mu$  under  $\prec_{\mathbb{L}_{\kappa, \omega}}$ ,  $\mathfrak{K}$ , then we can split  $\mathfrak{K}$  to  $\leq 2^{|T|^\kappa}$  classes each has a universal in  $\mu$ . Claim 1.7 below continues this.

Improving [Sh 552, 7.5]:

**1.1 Claim.** Assume  $\mu^+ < \lambda = \text{cf}(\lambda) < \mu^{\aleph_0}$  and  $A^* \subseteq {}^\omega \chi$ ,  $|A^*| \leq \mu$  and  $\mathbf{U}_{J_\chi^0 \upharpoonright A^*}(\lambda) < \mu^{\aleph_0}$ . Then in  $\mathfrak{K}_\lambda^{\text{tr}}$  there is no universal member where:

- (a)  $K_\lambda^{\text{tr}}$  is the class of  $(T, <)$ , trees with  $\omega + 1$  levels
- (b) embeddings preserve  $t <_T s$ ,  $\neg(t <_T s)$  and  $\text{lev}_T(x) = \alpha$  (for  $\alpha \leq \omega$ )
- (c)  $J_\chi^0 = \{B \subseteq {}^\omega \chi : \text{for some } m, k \text{ we have } (\forall \eta \in {}^m \chi)(\exists^{\leq k} \rho)(\eta \triangleleft \rho \in B)\}$
- (d)  $\mathbf{U}_J(\lambda) = \text{Min}\{|\mathcal{F}| : \mathcal{F} \text{ is a family of functions from } \text{Dom}(J) \text{ into } \lambda \text{ such that for any function } g \text{ from } \text{Dom}(J) \text{ to } \lambda \text{ for some } f \in \mathcal{F} \text{ the set } \{t \in \text{Dom}(J) : f(t) = g(t)\} \neq \emptyset \text{ mod}(J)\}$ , see [Sh:E12].

Inspite all cases dealt with in [Sh 552], there are still “missing” cardinals (see discussion in [Sh 622, §0]). Concerning  $\lambda$  singular  $2^{\aleph_0} < \mu^+ < \lambda < \mu^{\aleph_0}$ , by [Sh 622, 2.7t, 3.12t], [Sh:g] indicates that at least for most such cardinals there is no universal: as if  $\chi \in (\mu^+, \lambda)$  is regular, then  $\text{cov}(\lambda, \chi^+, \chi^+, \chi) < \mu^{\aleph_0}$ , (see [KjSh 455] on such proofs).

Let us mention concerning Case 1 (see from [Sh 622, §0]).

- 1.2 *Observation.* 1) If  $\lambda = \lambda^{\aleph_0}$  then in the class  $(\mathfrak{R}_\lambda^{\text{rtf}}, \leq_{\text{pr}})$ , defined below there is a universal one, in fact it is homogeneous universal.
- 2) If  $\lambda = \sum_{n < \omega} \lambda_n, \lambda_n = (\lambda_n)^{\aleph_0}$  then in  $(\mathfrak{R}_\lambda^{\text{rtf}}, \leq_{\text{pr}})$  there is a universal member (the parallel of special models for first order theories). (See Fuchs [Fu] on such abelian groups).
- 3) Similar results hold for  $\mathfrak{R}_{\bar{t}, \lambda}^{\text{rtf}}, \bar{t} = \langle t_\ell : \ell < \omega \rangle, 2 \leq t_\ell < \omega$ .

Recalling

- 1.3 Definition.** 1)  $\mathfrak{R}_\lambda^{\text{rtf}}$  is the class of torsion free abelian groups of cardinality  $\lambda$  ( $\leq_{\text{pr}}$  means pure subgroup).
- 2)  $\mathfrak{R}_{\bar{t}, \lambda}^{\text{rtf}}$  is the class of  $G \in \mathfrak{R}_\lambda^{\text{rtf}}$  such that there is no  $x \in G \setminus \{0\}$  divisible by  $\prod_{\ell < k} t_\ell$  for each  $k < \omega$  (where  $\bar{t} = \langle t_\ell : \ell < \omega \rangle, 2 \leq t_\ell < \omega$ ).
- 3) For  $G \in \mathfrak{R}_{\bar{t}, \lambda}^{\text{rtf}}$  let  $G^{[\bar{t}]}$  be the  $d_{\bar{t}}$ -completion where  $d_{\bar{t}} = d_{\bar{t}}[G]$  is the metric defined by  $d_{\bar{t}}(x - y) = \inf\{2^{-k} : \prod_{\ell < k} t_\ell \text{ divides } x - y \text{ in } G\}$ .

*Proof.* 1), 2) Follows from 3) with  $t_\ell = \ell!$

3) The point is

- (a) for  $G \in K_{\bar{t}}^{\text{rtf}}, G \leq_{\text{pr}} G^{[\bar{t}]} \in K_{\bar{t}}^{\text{rtf}}$  and  $G^{[\bar{t}]}$  has cardinality  $\leq \|G\|^{\aleph_0}$  and  $G^{[\bar{t}]}$  is  $d_{\bar{t}}$ -complete, remember  $G^{[\bar{t}]}$  is the  $d_{\bar{t}}$ -completion of  $G$ , it is unique up to isomorphism over  $G$ .  
Let  $\mathfrak{R}_{\bar{t}}^{\text{crtf}}$  is the class of  $d_{\bar{t}}$ -complete  $G \in \mathfrak{R}_{\bar{t}}^{\text{rtf}}$ .
- (b) If  $\langle G_\alpha : \alpha \leq \omega \rangle$  is  $\leq_{\text{pr}}$ -increasing sequence in  $\mathfrak{R}_{\bar{t}}^{\text{crtf}}$ , then for some  $G', n < \omega \Rightarrow G_n \leq_{\text{pr}} G' \leq_{\text{pr}} G_\omega, G'$  is the  $d_{\bar{t}}$ -completion of  $\bigcup_{n < \omega} G_n$
- (c)  $(\mathfrak{R}_{\bar{t}}^{\text{crtf}}, \leq_{\text{pr}})$  has amalgamation and the Löwenheim-Skolem property down to  $\lambda$  for any  $\lambda = \lambda^{\aleph_0}$
- (d) for each  $G \in \mathfrak{R}_{\bar{t}, \lambda}^{\text{rtf}}$ , we can find  $\langle (G_i, x_i) : i < \lambda^{\aleph_0} \rangle$  such that:
- (i)  $G \leq_{\text{pr}} G_i, x_i \in G_i \in \mathfrak{R}_{\bar{t}, \lambda}^{\text{crtf}}$
  - (ii) if  $G \leq_{\text{pr}} G', x \in G'$  then we can find  $i < \lambda^{\aleph_0}$  and a pure embedding  $h$  of  $G'$  into  $G_i \cap i, h \upharpoonright G = \text{the identity}, h(x) = x_i$  (and  $h''(G_i) \leq_{\text{pr}} G$ ).

The result now follows.

□<sub>1.2</sub>

(See more in [Sh 300, Ch.II,§3]). This holds also for  $\mathfrak{K}_\lambda^{\text{rs}(p)}$ .

We may wonder whether the existence result of 1.2 holds for a stronger embeddability notion. A natural candidate is

**1.4 Definition.** Let  $G_0 \leq_{\bar{t}} G_1$  if:  $G_0, G_1$  are abelian groups on which  $\| - \|_{\bar{t}}$  is a norm,  $G_0 \leq_{\text{pr}} G_1$  and  $G_0$  is a  $d_{\bar{t}}$ -closed subset of  $G_1$ .

We prove below that for the cases we are looking at, the answer is positive, but cardinals like  $\beth_\omega^+ < (\beth_\omega)^{\aleph_0}$  remains open.

*1.5 Fact.* 1) If  $\lambda$  is strong limit,  $\aleph_0 = \text{cf}(\lambda) < \lambda$ , then there is a universal member in  $(\mathfrak{K}_{\bar{t}, \lambda}^{\text{rtf}}, <_{\bar{t}})$  where  $\bar{t} = \langle t_\ell : \ell < \omega \rangle, 2 \leq t_\ell < \omega$ .

2) The same holds for  $(\mathfrak{K}_\lambda^{\text{rs}(p)}, \leq_{\langle p: \ell < \omega \rangle})$  where

**1.6 Definition.**  $\mathfrak{K}_\lambda^{\text{rs}(p)}$  is the class of abelian  $p$ -groups which are reduced and separable.

*Proof.* Let  $\lambda_n < \lambda_{n+1} < \lambda = \sum \lambda_n$  and  $2^{\lambda_n} < \lambda_{n+1}$ . The idea in both cases is to analyze  $M \in K_\lambda$  as the union of increasing chain  $\langle M_n : n < \omega \rangle, M_n \prec_{\mathbb{L}_{\lambda_n^+, \lambda_n^+}} M$ ,  $\|M_n\| = 2^{\lambda_n}, \lambda_n < \lambda$ .

Case 1:  $(\mathfrak{K}_{\bar{t}}^{\text{rtf}}, \leq_{\bar{t}})$ .

Specifically, we shall apply 1.7, 1.9 below with:

$$\mathfrak{K} = K^{\text{rtf}}$$

$$\mu_n = (2^{\lambda_n})^+$$

$$\leq_1 = \leq_0 \text{ is : } M_1 \leq_1 M_2 \text{ iff } (M_1, M_2 \in \mathfrak{K} \text{ and}) \\ M_1 \leq_{\bar{t}} M_2$$

$$\leq_2 \text{ is : } M_1 \leq_2 M_2 \text{ iff } M_1 \leq_1 M_2 \text{ and}$$

$$M_1 \prec_{L_{\aleph_1, \aleph_2}} M_2, \text{ or just :}$$

$$\text{if } G_1 \subseteq M_1, G_1 \subseteq G_2 \subseteq M_2,$$

and  $G_2$  is countable then there is an embedding  $h$  of  $G_2$  into  $M_1$  over  $G_1$ .

We should check the conditions in 1.7 which we postpone.

Case 2:  $(\mathfrak{K}^{\text{rs}(p)}, \leq_{\langle p: \ell < \omega \rangle})$  so  $\bar{t} = \langle p : \ell < \omega \rangle$ .

We let

$$\mathfrak{K} = \mathfrak{K}^{\text{rs}(p)}$$

$$\leq_1 = \leq_0 \text{ is } M_1 \leq_1 M_2 \text{ iff } (M_1, M_2 \in \mathfrak{K}^{\text{rs}(p)}, M_1 \leq_{\text{pr}} M_2 \text{ and}) M_1 \leq_{\bar{t}} M_2$$

$$\leq_2 \text{ as in Case 1.}$$

We shall finish the proof below.

**1.7 Claim.** *Assume*

- (a)  $\mathfrak{K}$  is a class of models of a fixed vocabulary close under isomorphism,  
 $\mathfrak{K}_\lambda \neq \emptyset$
- (b)  $\lambda = \sum_{n < \omega} \mu_n, \mu_n < \mu_{n+1}, 2^{\mu_n} < \mu_{n+1}, \mu_n$  is regular and the vocabulary of  $\mathfrak{K}$   
has cardinality  $< \mu_0$ .
- (c)  $\leq_1$  is a partial order on  $\mathfrak{K}$ , (so  $M \leq_1 M$ ) preserved under isomorphisms,  
and if  $\langle M_i : i < \delta \rangle$  is  $\leq_1$ -increasing and continuous then  $M_\delta = \bigcup_{i < \delta} M_i \in \mathfrak{K}$   
and  $i < \delta \Rightarrow M_i \leq_1 M_\delta$ .
- (d)  $\leq_2$  is a two-place relation on  $\mathfrak{K}$ , preserved under isomorphisms such that:  
if  $M \in \mathfrak{K}_\lambda$  then we can find  $\langle M_n : n < \omega \rangle$  such that:  $M_n \in \mathfrak{K}_{<\mu_n}, M_n <_2$   
 $M_{n+1}$  and  $M = \bigcup_{M < \omega} M_n$
- (e) If  $M_0 \in \mathfrak{K}_{<\mu_n}, M_0 \leq_1 M_1 \in \mathfrak{K}_{<\mu_{n+2}}, N^1 \leq_2 N^2 \in \mathfrak{K}_{<\mu_{n+1}}, h^1$  an isomor-  
phism from  $N^1$  onto  $M_0$ , then we can find  $M \in \mathfrak{K}_{<\mu}$  such that  $M_1 \leq_1 M$   
and there is an embedding  $h^2$  of  $N^2$  into  $M$  extending  $h^1$  satisfying  
 $h(N) \leq_1 M$ .

Then we can find  $\langle M_n^\alpha : n \leq \omega \rangle$  for  $\alpha < 2^{<\mu_0}$  such that:

- ( $\alpha$ )  $M_n^\alpha \in \mathfrak{K}_{<\mu_n}, M_n^\alpha \leq_1 M_{n+1}^\alpha, M_\omega^\alpha = \bigcup_{n < \omega} M_n^*$
- ( $\beta$ ) if  $M \in \mathfrak{K}_\lambda$  and the sequence  $\langle M_n : n < \omega \rangle$  is as in clause (d) then for some  
 $\alpha < 2^{<\mu_0}$  we can find an embedding  $h$  of  $M$  into  $M_\omega^\alpha$  such that  $h(M_n) \leq_1$   
 $M_{n+2}^\alpha$ .

*Proof.* Let

$$\begin{aligned} \mathfrak{K}'_0 = \{ & M : M \in \mathfrak{K} \text{ has universe an ordinal} \\ & < \mu_0, \text{ and there is } \langle M_n : n < \omega \rangle \text{ as in clause (c)} \\ & \text{with } M_0 \cong M \}. \end{aligned}$$

Clearly  $K'_0$  has cardinality  $\leq 2^{<\mu_0}$ , and let us list it as  $\langle M_0^\alpha : \alpha < \alpha^* \rangle$  with  $\alpha^* \leq 2^{<\mu_0}$ . We now choose, for each  $\alpha < \alpha^*$ , by induction on  $n < \omega$ ,  $M_n^\alpha$  such that:

- (i)  $M_n^\alpha \in \mathfrak{K}$  has universe an ordinal  $< \mu_n$
- (ii)  $M_n^\alpha \leq_1 M_{n+1}^\alpha$
- (iii) if  $N^1 \leq_2 N^2$ ,  $N^1 \in K_{<\mu_n}$ ,  $N^2 \in \mathfrak{K}_{<\mu_{n+1}}$ ,  $h^1$  is an embedding of  $N^1$  into  $M_{n+1}^\alpha$  satisfying  $h^1(N^1) \leq_1 M_{n+1}^\alpha$  then we can find  $h^2$  an embedding of  $N^2$  into  $M_{n+2}^\alpha$  extending  $h^1$  such that  $h^2(N^2) \leq_1 M_{n+2}^\alpha$ .

For  $n = 1$  we do not have much to do. (Use  $M_0^\alpha$  or  $M_1$  if  $\langle M_n : n < \omega \rangle$  is as in clause (c),  $M_0 \cong M_0^\alpha$  and use  $M_1^\alpha$  such that  $(M_1, M_0) \cong (M_1^\alpha, M_0^\alpha)$ ). For  $n + 1$ , let  $\{(h_{n,\zeta}^1, N_{n,\zeta}^1, N_{n,\zeta}^2) : \zeta < \zeta_n^*\}$  where  $\zeta_n^* \leq 2^{<\mu_{n+1}}$  list the cases of clause (iii) that need to be taken care of, with the set of elements of  $N_{n,\zeta}^2$  being an ordinal. Choose  $\langle N_{n+1,\zeta} : \zeta \leq \zeta_n^* \rangle$  which is  $\leq_1$ -increasing continuous satisfying  $N_{n+1,\zeta} \in \mathfrak{K}_{<\mu_{n+2}}$ . We choose  $N_{n+1,\zeta}$  by induction on  $\zeta$ . Let  $N_{n+1,0} = N_{n+1}$ , for  $\zeta$  limit let  $N_{n+1,\zeta} = \bigcup_{\xi < \zeta} N_{n+1,\xi}$  and use clause (b).

Lastly, for  $\zeta = \xi + 1$  use clause (d) of the assumption with  $h_{n,\zeta}^1(N_{n,\xi}^1)$ ,  $N_{n+1,\xi}$ ,  $N_{n,\xi}^1$ ,  $N_{n,\xi}^2$ ,  $h_{n,\xi}^1$  here standing for  $M_0, M_1, N^1, N^2, h^1, h^2$  there.  $\square_{1.7}$

*1.8 Remark.* 1) We can choose  $\langle M_0^\alpha : \alpha < \alpha^* \rangle$  just to represent  $\mathfrak{K}'_0$ , and similarly later (and so ignore the “with the universe being an ordinal”).

2) Actually, the family of  $\langle M_n : n < \omega \rangle$  as in clause (c) such that  $M_n$  has set of elements an ordinal, forms a tree  $T$  with  $\omega$  levels with the  $n$ -th level having  $\leq 2^{<\mu_n}$  members, and we can use some free amalgamations of it. This gives a variant of 1.7.

3) We can put into the axiomatization the stronger version of (d) from 1.7 proved in the proof of 1.5 so we can weaken  $(\beta)$  of 1.9 below.

4) E.g., in (d) we can add  $M_n <_* M$  and so weaken clause  $(\beta)$  of 1.7.

*1.9 Conclusion.* In 1.7 there is in  $\mathfrak{K}_\lambda$  a universal member under  $\leq_0$ -embedding if in addition



- ( $\alpha$ ) for any  $< \lambda$  (or just  $\leq 2^{<\mu_0}$ ) members we can find one member into which all of them are embeddable
- ( $\beta$ ) if  $M_n \leq_1 M_{n+1}, M_n \leq_1 N_n, N_n \leq_2 N_{n+1}$  and  $M_n \in K_{<\mu_{n+2}}, N_n \in K_{<\mu_{n+1}}, M = \cup\{M_n : n < \omega\}$  for  $n < \omega$  then  $\bigcup_{n<\omega} M_n \leq_0 \bigcup_{n<\omega} N_n$ .

*Proof.* Easy.

### Continuation of the proof of 1.5

We have to check the demands in 1.9 and 1.7.

Case 1: The least trivial clause to check is (d)

Clause (d): (non-symmetric amalgamation)

Without loss of generality  $h_1$  = the identity,  $N^1 \cap M_1 = M_0 = N_0$ . Just take the free amalgamation  $M = N^1 *_{M_0} M_1$  (in the variety of abelian groups) and note that naturally  $M_1 \leq_1 M$ .

Case 2: Similar.  $\square_{1.5}$

\* \* \*

*1.10 Discussion.* 1) Can we in 1.7, 1.9 replace  $\text{cf}(\lambda) = \aleph_0$ , by  $\text{cf}(\lambda) = \theta > \aleph_0$ ? If increasing union of chains in  $K_{<\lambda}$  behaves nicely then yes, with no real problem. More elaborately

- (i) in 1.7(c), we get  $\langle M_\varepsilon : \varepsilon < \theta \rangle$  such that  $M_\varepsilon \in K_{<\mu_\varepsilon}, \langle M_\varepsilon : \varepsilon < \theta \rangle$  is  $\subseteq$ -increasing continuous,  $M_\varepsilon <_2 M_{\varepsilon+1}, M = \cup\{M_\varepsilon : \varepsilon < \theta\}$
- (ii) we add: if  $\langle M_i : i \leq \delta \rangle$  is  $\leq_1$ -increasing continuous,  $M_i \in K_{<\lambda}$  and  $i < \delta \Rightarrow M_i \leq_1 N$  then  $M_\delta \leq_i N$ .

Otherwise we seem to be lost.

2) Suppose  $\lambda = \sum_{n<\omega} \lambda_n, \lambda_n = (\lambda_n)^{\aleph_0} < \lambda_{n+1}$ , and  $\mu < \lambda_0, \lambda < 2^\mu$  (i.e., Case 6b of

[Sh 622]). Is there a universal member in  $(\mathfrak{K}_{\bar{t}, \lambda}^{\text{rtf}}, <_{\bar{t}})$ ?

Assume  $\mathbf{V} \models “\mu = \mu^{<\mu}, \mu < \chi”$  and  $\mathbb{P}$  is the forcing notion of adding  $\chi$  Cohen subsets to  $\mu$  (that is  $\mathbb{P} = \{f : f \text{ a partial function from } \chi \text{ to } 2, |\text{Dom}(f)| < \mu\}$  ordered by inclusion). Do we have in  $\mathbf{V}^{\mathbb{P}} : \lambda < \lambda^{\aleph_0} \ \& \ \mu < \lambda < \chi \Rightarrow$  in  $(\mathfrak{K}_{\bar{t}, \lambda^+}^{\text{rtf}}, <_{\bar{t}})$  there is no universal member.

Maybe continuing [Sh:e, Ch.III, §2] we can get consistency of the existence.

3) Now if  $\lambda = \lambda^{\aleph_0}$  then in  $(\mathfrak{K}_\lambda^{\aleph_1\text{-free}}, \subseteq)$  there is no universal member; see [Sh:e,

Ch.IV], because amalgamation fails badly. Putting together there are few cardinals which are candidates for consistency of existence. In (2), if there is a regular  $\lambda' \in (\mu, \lambda)$  with  $\text{cov}(\lambda, \lambda^+, \lambda^+, \lambda') < 2^\mu$  then contradict 1.2.

4) Considering consistency of existence of universal in (2), it is natural to try to combine the independent results in [Sh:e, Ch.IV] and [DjSh 614].

## §2 THE CLASS OF GROUPS

We know ([ShUs 789]) that the class of groups has  $\text{NSOP}_4$  and  $\text{SOP}_3$  (from [Sh 500, §2]). We shall prove two results on the place of the class of groups in the model theoretic classification. One says that it falls in “the complicated side” for some division: it has the oak property ([DjSh 710]). Well the case there was complete first order theories. Its meaning (and “no universal” consequences) are clear in a more general context, see below. Amenability is a condition on a theory (or class) which gives sufficient condition for existence of universals and in suitable models of set theory (see [DjSh 614]). We prove that the class of groups satisfies it; so for this division the class of groups falls in the non-complicated side.

**2.1 Claim.** *The class of groups has the oak property by some quantifier free formula.*

**2.2 Definition.**

- (1) A theory  $T$  is said to satisfy the oak property as exhibited by (or just by) a formula  $\varphi(\bar{x}, \bar{y}, \bar{z})$  iff for any  $\lambda, \kappa$  there are  $\bar{b}_\eta (\eta \in {}^\kappa\lambda)$  and  $\bar{c}_\nu (\nu \in {}^\kappa\lambda \text{ and } \bar{a}_i (i < \kappa))$  such that
  - (a)  $[\eta \in \nu, \nu \in {}^\kappa\lambda]$  implies  $\varphi[\bar{a}, \ell g(\eta), \bar{b}_\eta, \bar{c}_\nu]$
  - (b) if  $\eta \in {}^\kappa>\lambda$  and  $\eta\langle\alpha\rangle \in \nu_1 \in {}^\kappa\lambda$  and  $\eta\langle\beta\rangle \in \nu_2 \in {}^\kappa\lambda$ , while  $\alpha \neq \beta$  and  $i > \ell g(\eta)$ , then  $\neg \exists \bar{y} [\varphi(\bar{a}_i, \bar{y}, \bar{c}_{\nu_1}) \wedge \varphi(\bar{a}_i, \bar{y}, \bar{c}_{\nu_2})]$  and in addition  $\varphi$  satisfies
  - (c)  $\varphi(\bar{x}, \bar{y}_1, \bar{z}) \wedge \varphi(\bar{x}, \bar{y}_2, \bar{z})$  implies  $\bar{y}_1 = \bar{y}_2$ .

- (2) A theory  $T$  has the  $\Delta$ -oak property if it is exhibited by some  $\varphi(\bar{x}, \bar{y}, \bar{z}) \in \Delta$ .

*Proof.* Let  $w(x, y)$  be complicated enough words, say of length  $k^* = 100$ .

For  $\kappa, \lambda$  let  $G = G_{\lambda, \kappa}$  be

Version A: The groups generated by  $\{x_i : i < \kappa\} \cup \{y_\eta : \eta \in {}^\kappa>\lambda\} \cup \{z_\nu : \nu \in {}^\kappa\lambda\}$  freely except the set of equations

$$\Gamma = \{y_{\nu \upharpoonright i} = w(z_\nu, x_i) : \nu \in {}^\kappa\lambda, i < \kappa\}.$$

We have to show that

- (\*) if  $\nu \in {}^\kappa\lambda, i < \kappa, \rho \in {}^i\lambda \setminus \{\nu \upharpoonright i\}, G \models y_\rho \neq w(z_\nu, x_i)$ .

Now

- (\*) each works  $y_{\nu \upharpoonright i}^{-1} w(z_\nu, x_i)$ ,  $i$  is cyclically reduced
- (\*\*) for any two such words or cyclical permutations of them which are not equal, any common segment has length  $< k^*/6$ .

Explanation and why this is enough see [LySc77].

Version B:  $\Gamma = \{y_{\nu \upharpoonright i} = [z_\nu, x_i] : \nu \in {}^\kappa\lambda, i < \kappa\}$  is O.K. (and simpler).

Why? Let  $G_1$  be the group generated by

$$Y = \{x_i : i < \kappa\} \cup \{z_\nu : \nu \in {}^\kappa\mu\}$$

freely except the set of equation  $\Gamma_2 = \{[z_\nu, x_i] = ([z_\eta, x_i])^{-1} : i < \kappa, \nu \in {}^\kappa\lambda, \eta \in {}^\kappa\lambda\}$ . So for  $i < \kappa, \rho \in {}^i\lambda$  we can choose  $y_\rho \in G_1$  such that  $\eta \in {}^\kappa\lambda, \eta \upharpoonright i = \rho \Rightarrow$  and  $\nu \upharpoonright i = \eta \upharpoonright i, y_\rho = [z_\eta, x_i]$ . Clearly it suffices to prove

$$\odot \text{ in } G_1/N \text{ if } \nu, \eta \in {}^i\lambda \text{ \& } i < \kappa \text{ then } [[z_\nu, x_i] = [z_\eta, x_i] \Leftrightarrow \nu \upharpoonright i = \eta \upharpoonright i.$$

The implication  $\Leftarrow$  holds trivially. For the other direction let  $j < \kappa$  and  $\eta, \nu \in {}^\kappa\lambda$  be such that  $\eta \upharpoonright j \neq \nu \upharpoonright j$  and we shall prove that  $G \models y_{\eta \upharpoonright j} \neq y_{\nu \upharpoonright j}$ .

Let  $N_*^*$  be the normal subgroup of  $G_1$  generated by

$$\begin{aligned} & \{x_i : i \in \kappa \text{ \& } i \neq j\} \cup \{z_\rho : \rho \in {}^\kappa\lambda, \rho \upharpoonright j \notin \{\eta \upharpoonright j, \nu \upharpoonright j\}\} \\ & \cup \{z_\rho z_\eta^{-1} : \rho \in {}^\kappa\lambda, \rho \upharpoonright j = \eta \upharpoonright j\} \\ & \cup \{z_\rho z_\nu^{-1} : \rho \in {}^\kappa\lambda, \rho \upharpoonright j = \nu \upharpoonright j\}. \end{aligned}$$

Clearly  $N_*$  includes  $N$  and  $G_1/N_*$  is generated by  $\{x_i\} \cup \{z_\eta, z_\nu\}$  freely, hence  $G_1/N \models [z_\eta, x_i] \neq [z_\nu, x_i]$  hence  $G_1 \models y_{\eta \upharpoonright j} \neq y_{\nu \upharpoonright j}$  as required.  $\square_{2.1}$

**2.3 Definition.** Let  $\lambda = \text{cf}(\lambda) > \aleph_0$  we define  $K_{\text{ap}} = (K_{\text{ap}}, \leq_{K_{\text{ap}}})$  (see [Sh 457], [DjSh 614]):

- (A)  $G \in K_{\text{ap}}$  iff:
  - (a)  $G$  is a group
  - (b)  $\delta \in S_\lambda^{\lambda^+} \Rightarrow G \cap \delta$  is a subgroup of  $G$  (this implies  $0 < \delta < \lambda^+$  \&  $\lambda(\delta) \Rightarrow G \cap \delta$
  - (c)  $e_G = 0$
- (B)  $G_1 \leq_{\text{AP}} G_2$  iff  $G_1 \subseteq G_2$  (i.e., subgroup both in  $K_{\text{ap}}$ )

**2.4 Claim.**  $K_{\text{ap}}$  is a  $\lambda$ -approximation family.

*Proof.* Main Point: Amalgamation.

So we have  $G_0 \subseteq G_\ell$  for  $\ell = 1, 2$  which are groups from  $K_{\text{ap}}$  and without loss of generality  $G_1 \cap G_2 = G_0$ . Let  $G_3$  be the free amalgamation of  $G_1, G_2$  over  $G_0$  so ([LySc77])

(\*) so  $G_\ell \subseteq G$  (and still  $G_1 \cap G_2 = G_0$ ).

For  $\delta \in S_\lambda^{\lambda^+}$ ,  $\ell < 3$  let  $G_{\ell,\delta} = G_\ell \cap \delta$  and let  $G_{3,\delta} = \langle G_\delta^1 \cup G_\delta^2 \rangle_{G_3}$ .  
We shall show that

⊗  $G_\ell \cap G_{3,\delta} = G_{\ell,\delta}$  for  $\ell < 3$ .

This suffices as it implies that there is a one to one mapping  $h$  from  $G_3$  into  $\lambda^+$  such that  $\delta \in S_\lambda^{\lambda^+}$  &  $x \in G_{3,\delta} \Rightarrow g(x) < \delta$  hence  $\delta < \lambda^+$  &  $\lambda/\delta$  &  $x \in G_{3,\delta} \Rightarrow g(x) < \delta$ . So  $h''(G_{3,\delta})$  is a common upper bound.

*Proof of ⊗.* Fix  $\delta$  and obviously  $G_{1,\delta} \cap G_{2,\delta} = G_{0,\delta}$ .

Let  $z \in G_{3,\delta} \setminus G_{1,\delta} \setminus G_{2,\delta}$  and we shall prove  $z \notin G_1 \cup G_2$ , by (\*) this suffices so  $z$  is a product of members of  $G_{1,\delta} \cup G_{2,\delta}$  so by the rewriting process we can find  $n < \omega$  and  $x_k \in G_1, y_k \in G_2$  (for  $k < n$ ) such that:

- (a)  $z = x_0 y_0 x_1 y_1 \dots x_n y_n$  in the group  $G_3$
- (b)  $k > 0 \Rightarrow x_k \in G_{1,\delta} \setminus G_{0,\delta}$
- (c)  $k < n \rightarrow y_k \in G_{2,\delta} \setminus G_{0,\delta}$
- (d)  $x_0 \in G_{1,\delta} \setminus G_{0,\delta}$  or  $x_0 = e$
- (e)  $y_n \in G_{2,\delta} \setminus G_{0,\delta}$  or  $y_n = e$ .

Toward contradiction assume  $z \in G_1 \cup G_2$ , so  $z \in (G_1 \setminus G_{1,\delta}) \cup (G_2 \setminus G_{2,\delta})$  so by symmetry without loss of generality  $z \in G_1 \setminus G_{1,\delta}$ . If  $y_n \neq e$ , then by computations the word  $w = x_0 y_0 \dots x_n y_n z^{-1}$  is equal to  $e$  but it in canonical form ([LySc77]), this is a contradiction.

If  $y_n = e, x_n z^{-1} \notin G_0$  we get similar contradiction using the word  $x_0 y_0 \dots x_{n-1} (x_n z^{-1})$ . So assume  $x_n z^{-1} \in G_0$ , if  $n \geq 1$  then use the word  $x_0 y_0 \dots x_{n-1} (y_{n-2} x_n z^{-1})$ . We remain with the case  $n = 0, y_n = e, x_n z^{-1} \in G_0$ , but then  $z = x_0 y_0 = x_0 \in G_{1,\delta}$ , contradiction. □<sub>2.4</sub>

**2.5 Claim.**  $K_{\text{ap}}$  is simple.

*Proof.* So assume

- (\*) (a)  $\delta_0 < \delta_1$
- (b)  $M_{\delta_\ell} \leq_{K_{\text{ap}}} N_{\delta_\ell}, N_{\delta_0} \subseteq \delta_1$
- (c)  $M_{\delta_\ell} \leq_{K_{\text{ap}}} M$
- (d)  $N_{\delta_1} \cap N_{\delta_2} = N \subseteq \delta_0$
- (e)  $h$  is a lawful isomorphism from  $N_{\delta_1}$  onto  $N_{\delta_2}$  which is the identity on  $N$
- (f)  $N_{\delta_\ell} \cap M = M_{\delta_\ell}$
- (g)  $h$  maps  $M_{\delta_1}$  onto  $M_{\delta_2}$

We should find a common  $\leq_{K_{\text{ap}}}$ -upper bound to  $N_{\delta_1}, N_{\delta_2}, M$ .

Now by the theory of HNN extensions there is a group  $G_1$  extending  $M$  and  $z \in G_1$  such that conjugating by  $z$  induce  $h \upharpoonright M_{\delta_0}$ ,  $G = \langle M, y \rangle_G$  and  $G_1$  is extending by  $z$  freely except the equations  $zxz^{-1} = h(x)$  for  $x \in M_{\delta_0}$ .

Let  $G_0 = \langle M_{\delta_0}, z \rangle_{G_1}$ , easily

- ⊗  $G_0$  is the free extension of  $M_{\delta_0}$  by  $z$ .

Let  $H_0 = \langle z \rangle_{G_0}$ ,  $H_1$  be the free product of  $N$  and  $H_0$  and  $G_2$  be the free product of  $H_1, N_{\delta_0}$  over  $N$ . Let  $g$  be the identity map on  $G_0$ , it is an embedding  $G_0$  into  $G_1$  extending  $\text{id}_{M_{\delta_0}}$  and mapping  $z$  to itself.

So without loss of generality

- (\*)  $G_0 \subseteq G_1, G_1 \cap G_2 = 0$ .

Let  $G_3$  be the free amalgamation of  $G_1, G_2$  over  $G_0$  and we define an embedding  $f$  of  $N_{\delta_1}$  into  $G$  recalling  $\text{Rang}(h^{-1}) \subseteq N_{\delta_0} \subseteq G_2 \subseteq G$ :

$$x \in N_{\delta_2} \Rightarrow f(x) = z(h^{-1}(x))z^{-1}.$$

So to  $G_3$  the groups  $M, N_{\delta_0}, N_{\delta_1}$  are embedded with no identification. We still have to show as in the proof of 2.4 that no undesirable identification occurs, that is

- ⊗ if  $\delta \in S_\lambda^{\lambda^+}, z_1 \in \langle (N_{\delta_0} \cup N_{\delta_1} \cup M) \cap \delta \rangle_{G_3}, z_2 \in (N_{\delta_1} \cup N_{\delta_2} \cup M) \setminus \delta$  then  $G_3 \models z_1 \neq z_2$ .

Case 1:  $\delta \leq \delta_1$ .

In this case, as  $(N_{\delta_0} \cup N_{\delta_1} \cup M) \cap \delta \subseteq (N_{\delta_0} \cap \delta) \cup (N_{\delta_1} \cap \delta) \cup (M \cap \delta) \subseteq N_{\delta_0} \cap \delta$ , clearly  $z_1$  belongs to  $N_{\delta_0} \cap \delta$ ; now if  $z_2 \in N_{\delta_1}$  this is totally trivial and if  $z_2 \notin N_{\delta_1}$ ,  $z_2 \in M \cup N_{\delta_1} \setminus \delta_1$  we follow the amalgamations.

Case 2:  $\delta \geq \delta_1$ .

Let  $\delta'_1 = \delta, \delta'_0 \in S_{\lambda}^{\lambda^+}$  be such that  $h$  maps  $N_{\delta_0} \cap \delta'_0$  onto  $N_{\delta_1} \cap \delta'_1$ . We can repeat the construction above for  $h' = h \upharpoonright N_{\delta_0} \cap \delta'_0, N'_{\delta_0} = N_{\delta_0}$  (not  $N_{\delta_0} \cap \delta'_0$ !)  $N'_{\delta_1} = N_{\delta_1} \cap \delta'_1, M' = M \cap \delta'_1, N' = N$  and on each stage show that we get a subgroup of the group we get in the original version.

*2.6 Conclusion.* The class of groups is amenable (see [DjSh 710, Definition 0.1]).

## §3 ON THE OAK PROPERTY

We can in the “no universal” results in [DjSh 710] deal also with the case of singular cardinal.

**3.1 Claim.** *Assume*

- (a) *T is a complete first order theory with the oak property*
- (b) (i)  $\kappa = \text{cf}(\mu) \leq \sigma < \mu < \lambda = \text{cf}(\lambda) \leq \lambda_1 \leq \lambda_2$   
(ii)  $\kappa \leq \sigma \leq \lambda_1, |T| \leq \lambda_2$   
(iii)  $\mu^\kappa \geq \lambda_2$
- (c) *let  $\bar{C} = \langle C_\delta : \delta \in S \rangle, C_\delta \subseteq \delta, \text{otp}(C_\delta) = \mu, S \subseteq \lambda$  stationary,  $J =: \{A \subseteq \lambda : \text{for some club } E \text{ of } \lambda, \delta \in S \cap A \Rightarrow C_\delta \not\subseteq E\}, \lambda \notin J$  and  $\alpha < \lambda \Rightarrow \lambda > |\{C_\delta \cap \alpha : \alpha \in \text{nacc}(C_\delta), \delta \in S\}|$*
- (d)  $\mathbf{U}_J(\lambda_1) < \lambda_2$
- (e) *for some  $\mathcal{P}_1, \mathcal{P}_2$  we have*
  - (i)  $\mathcal{P}_1 \subseteq [\lambda_1]^\kappa, \mathcal{P}_2 \subseteq [\sigma]^\kappa$
  - (ii) *if  $g : \sigma \rightarrow \lambda_1$  is one to one then for some  $X \in \mathcal{P}_2$ , we have  $\{g(i) : i \in X\} \in \mathcal{P}_1$*
  - (iii)  $|\mathcal{P}_1| \leq \lambda_2, \lambda_1$
  - (iv)  $|\mathcal{P}_2| \leq \lambda_1$ .

Then  $\text{univ}(\lambda_1, T) \geq \lambda_2$ .

Recall

- 3.2 Definition.** 1) For  $\bar{N} = \langle N_\gamma : \gamma < \lambda \rangle$  an elementary-increasing continuous sequence of models of  $T$  of size  $< \lambda$  and for  $a, c \in N_\lambda = \bigcup_{\gamma < \lambda} N_\gamma$  and  $\delta \in S$ , we let
- $$\text{inv}_{\bar{N}}(c, C_\delta, a) = \{\zeta < \mu : (\exists b \in N_{\alpha(\delta, \zeta+1)} \setminus N_{\alpha(\delta, \zeta)}) N_\lambda \models \varphi[a, b, c]\}.$$
- 2) For a set  $A$  and  $\delta, \bar{N}$  as above, let  $\text{inv}_{\bar{N}}^A(c, C_\delta) = \bigcup \{\text{inv}_{\bar{N}}(c, C_\delta, a) : a \in A\}.$

*Proof.*

Step A: Assume toward contradiction  $\theta =: \text{univ}(\lambda_1, T) < \lambda_2$ , so let  $\langle N_j^* : j < \theta \rangle$  exemplifies this and  $\theta_1 = \theta + \lambda_1 + |T| + \mathcal{U}_J(\lambda_1)$ .

Without loss of generality the universe of  $N_j^*$  is  $\lambda_1$ .



Step B: By the definition of  $\mathbf{U}_J(\lambda_1)$  there is  $\mathcal{A}$  such that:

- (a)  $\mathcal{A} \subseteq [\lambda_1]^\lambda$
- (b)  $|\mathcal{A}| \leq \mathbf{U}_J(\lambda_1)$
- (c) if  $f : \lambda \rightarrow \lambda_1$  then for some  $A \in \mathcal{A}$  we have  $\{\delta \in S : f(\delta) \in A\} \neq \emptyset \bmod J$ .

For each  $X \in \mathcal{P}_1, j < \theta$  and  $A \in \mathcal{A}$  let  $M_{j,X,A}$  be an elementary submodel of  $N_j^*$  of cardinality  $\lambda$  which includes  $X \cup A \subseteq \lambda_1$ , and let  $\bar{M}_{j,X,A} = \langle M_{j,X,A,\varepsilon} : \varepsilon < \lambda \rangle$  be a filtration of  $M_{j,X,A}$ .

Lastly, consider

$$\mathcal{B} = \{\text{inv}_{M_{j,X,A}}^X(a, C_\delta) : j < \theta, X \in \mathcal{P}_1, A \in \mathcal{A}, \delta \in S \text{ and } a \in M_{j,X,A}\}.$$

Step C: Easily we have  $|\mathcal{B}| \leq \theta_1 < \lambda_2$ , hence there is  $B^* \in [\mu]^\kappa \setminus \mathcal{B}$ . Now let  $M^*$  be a  $\lambda^+$ -saturated model of  $T$ , in which  $a_i, b_\eta (\eta \in {}^{\kappa>}(\lambda_2)), c_\nu (\nu \in {}^\kappa(\lambda_2)), \varphi$  are as in the definition of the oak property and for each  $Y \in \mathcal{P}_2$ , choose  $\langle N_{Y,\varepsilon} : \varepsilon < \lambda \rangle, \langle c_{Y,\varepsilon,\delta} : \delta \in S \rangle$  as in 3.2.

Let  $N \prec M^*, \|N\| = \lambda_1$  such that  $\{a_i : i < \sigma\} \cup \bigcup \{N_{y,\varepsilon} : y \in \mathcal{P}_2, \varepsilon < \lambda\} \subseteq N$ .

Step D: By our choice of  $\langle N_j^* : j < \sigma \rangle$ , there is  $j(*) < \theta$  and elementary embedding  $f : N \rightarrow N_{j(*)}^*$ . By an assumption there are  $Y \in \mathcal{P}_1$  such that  $\{f(a_i) : i \in Y\} = X \in \mathcal{P}_2$ . Also by the choice of  $\mathcal{A}$  there is  $A \in \mathcal{A}$  such that  $\{\delta \in S : f(a_{Y,\delta}) \in A\} \neq \emptyset \bmod D$ .

Now we can finish (note that we use here again the last clause in the definition of the oak property).

**3.3 Definition.** 1) The formula  $\varphi(x, y, z)$  has the oak' property in  $T$  (the first order complete theory) if: omitting clause (c) in [DjSh 710, 1.8].

2)  $T$  has the oak property if some  $\varphi(x, y, z)$  has it  $m^+$ .

**3.4 Claim.** *Assume*

- (a)  $T$  has the oak property,  $|T| \leq \lambda$
- (b)  $\bar{C} = \langle C_\delta : \delta \in S \rangle, J$  are as in (c) of 3.1.

Then for each  $B^* \in [\mu]^\kappa$ ,  $T$  has a model  $N^*$  of cardinal  $\lambda$  and  $\langle a_i : i < \kappa \rangle$  as in [DjSh 710, 1.8], satisfying

- ⊗ if  $N$  is a model of  $T$  of cardinal  $\lambda$  with filtration  $\bar{N} = \langle N_\alpha : \alpha < \lambda \rangle$  and  $f$  is an elementary embedding of  $N^*$  into  $N$  then

$\{\delta \in S : \text{for some } a \in N^* \text{ we have}$

$$B^* = \text{inv}_{\bar{N}}^{\{f(a_i) : i < \kappa\}}(C_\delta, a) = S \text{ mod } J.$$

*Proof.* As usual, there is  $N^* \models T$  with filtration  $\bar{N}^* = \langle N_i^* : i < \lambda \rangle$  and  $I \subseteq {}^\kappa \lambda$  of cardinality  $\lambda$ ,  $\langle b_\eta : \eta \in \mathcal{T} \rangle$  and  $\nu_\delta \in {}^\kappa(C_\delta) \cap \lim_\kappa(T)$  for  $\delta \in S$  and  $\langle c_{\nu_\delta} : \delta \in S \rangle$  such that

(a)  $\langle a_i : i < \kappa \rangle, \langle b_\eta : \eta \in \mathcal{T} \rangle, \langle C_{\nu_\delta} : \delta \in S \rangle$  are as in the Definition 3.3

(b)  $(\nu_\delta(i) \cap C_\delta) = (\text{the } i\text{-th member of } B^*) + 1.$

So let  $N, \langle N_\varepsilon : \varepsilon < \lambda \rangle, f$  be as in the claim. Without loss of generality the universes of  $N^*$  and  $N$  are  $x$ .

Let

$$E_* = \{\delta < \lambda : \delta \text{ limit, } f''(\delta) = \delta, |N_\delta| = \delta = |N_\delta^*| \text{ and } (N_\delta^*, N_\delta^*, f) \prec (N^*, N^*, f)\}$$

it is a club of  $\lambda$ . For each  $i < \kappa$  let

$$\begin{aligned} W_i = \{ \alpha : & \text{for some } \delta \in S, \alpha \in C_{\delta_1} \subseteq E, \nu_\delta(i) > \alpha, \\ & \text{but } \varphi(f(a_i), y, f(c_{\nu_\delta})) \text{ is satisfied (in } N) \\ & \text{by some } b \in N_\alpha \} \end{aligned}$$

- ⊗  $W_i$  is not stationary

[Why? Let  $\mathfrak{B} \prec (\mathcal{H}(\lambda^+), \in, <^*)$  be such that  $\bar{N}, \bar{N}^*, \langle a_\varepsilon : \varepsilon < \kappa \rangle, \langle b_\eta : \eta \in \mathcal{T} \rangle, \langle c_{\nu_\delta} : \delta \in S \rangle$  belong and  $\mathfrak{B} \cap \lambda = \alpha \in W_i$  and assume  $b \in \mathfrak{B} \cap \alpha, N \models \varphi(f(a_i), b, f(c_{\nu_\delta}))$ . So there is  $\delta' \in S \cap \delta$  such that  $N \models \varphi[f(a_1), b, f(c_{\nu_\delta})]$ . But  $\nu_\delta(i) \geq \alpha > \nu_{\delta'}(i)$  hence  $\varphi(a_i, y, c_{\nu_\delta}), \varphi(a_i, y, c_{\nu_{\delta'}})$  are incompatible (in  $N^*$ ) hence their images by  $f$  are incompatible in  $N$  by  $b$  satisfies both contradictions, so  $W_i$  is not stationary.]

So there is a club  $E^*$  of  $\lambda$  included in  $E_\kappa$  and disjoint to  $W_i$  for each  $i < \kappa$ . So there is  $\delta^* \in S$  such that  $C_\delta \subseteq E^*$  and we get contradiction as usual.

Question: Can we combine 3.1, 3.4?

(For many singular  $\lambda_2$ 's, certainly yes).

## REFERENCES.

- [DjSh 710] Mirna Džamonja and Saharon Shelah. On properties of theories which preclude the existence of universal models. *Annals of Pure and Applied Logic*, **submitted**. math.LO/0009078.
- [DjSh 614] Mirna Džamonja and Saharon Shelah. On the existence of universal models. *Archive for Mathematical Logic*, **accepted**. math.LO/9805149.
- [Fu] Laszlo Fuchs. *Infinite Abelian Groups*, volume I, II. Academic Press, New York, 1970, 1973.
- [GrSh 174] Rami Grossberg and Saharon Shelah. On universal locally finite groups. *Israel Journal of Mathematics*, **44**:289–302, 1983.
- [KjSh 409] Menachem Kojman and Saharon Shelah. Non-existence of Universal Orders in Many Cardinals. *Journal of Symbolic Logic*, **57**:875–891, 1992. math.LO/9209201.
- [KjSh 455] Menachem Kojman and Saharon Shelah. Universal Abelian Groups. *Israel Journal of Mathematics*, **92**:113–124, 1995. math.LO/9409207.
- [KoSh 492] Peter Komjath and Saharon Shelah. Universal graphs without large cliques. *Journal of Combinatorial Theory. Ser. B*, **63**:125–135, 1995.
- [LySc77] Roger C. Lyndon and Paul E. Schupp. *Combinatorial group theory*, volume 89 of *Ergebnisse der Mathematik und ihrer Grenzgebiete*. Springer-Verlag, Berlin–Heidelberg–New York, 1977.
- [Sh:E12] Saharon Shelah. Analytical Guide and Corrections to [Sh:g]. math.LO/9906022.
- [Sh:e] Saharon Shelah. *Non-structure theory*, accepted. Oxford University Press.
- [Sh 26] Saharon Shelah. Notes on combinatorial set theory. *Israel Journal of Mathematics*, **14**:262–277, 1973.
- [Sh 300] Saharon Shelah. Universal classes. In *Classification theory (Chicago, IL, 1985)*, volume 1292 of *Lecture Notes in Mathematics*, pages 264–418. Springer, Berlin, 1987. Proceedings of the USA–Israel Conference on Classification Theory, Chicago, December 1985; ed. Baldwin, J.T.
- [Sh 457] Saharon Shelah. The Universality Spectrum: Consistency for more classes. In *Combinatorics, Paul Erdős is Eighty*, volume 1, pages 403–420. Bolyai Society Mathematical Studies, 1993. Proceedings of the Meeting in honour of P.Erdős,

- Keszthely, Hungary 7.1993; A corrected version available as ftp://ftp.math.ufl.edu/pub/settheory/shelah/457.tex. math.LO/9412229.
- [Sh:g] Saharon Shelah. *Cardinal Arithmetic*, volume 29 of *Oxford Logic Guides*. Oxford University Press, 1994.
- [Sh 500] Saharon Shelah. Toward classifying unstable theories. *Annals of Pure and Applied Logic*, **80**:229–255, 1996. math.LO/9508205.
- [Sh 456] Saharon Shelah. Universal in  $(< \lambda)$ -stable abelian group. *Mathematica Japonica*, **43**:1–11, 1996. math.LO/9509225.
- [Sh 552] Saharon Shelah. Non-existence of universals for classes like reduced torsion free abelian groups under embeddings which are not necessarily pure. In *Advances in Algebra and Model Theory. Editors: Manfred Droste and Ruediger Goebel*, volume 9 of *Algebra, Logic and Applications*, pages 229–286. Gordon and Breach, 1997. math.LO/9609217.
- [Sh 622] Saharon Shelah. Non-existence of universal members in classes of Abelian groups. *Journal of Group Theory*, **4**:169–191, 2001. math.LO/9808139.
- [ShUs 789] Saharon Shelah and Alex Usvyatsov. Banach spaces and groups - order properties and universal models. *Israel Journal of Mathematics*, **submitted**. math.LO/0303325.